

# Time-ordered perturbation theory on non-commutative spacetime: Basic rules

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**Abstract.** Assuming that the  $S$ -matrix on non-commutative (NC) spacetime can still be developed perturbatively in terms of the time-ordered exponential of the interaction Lagrangian, we investigate the perturbation theory of NC field theory. We first work out with care some typical Green functions starting from the usual concepts of time-ordering and commutation relations for free fields. The results are found to be very different from those in the naive approach pursued in the literature. A simple framework then appears naturally which can incorporate the new features of our results and which turns out to be the usual time-ordered perturbation theory extended to the NC context. We provide the prescriptions for computing  $S$ -matrix elements and Green functions in this framework. We also emphasize that the naive seemingly covariant approach cannot be reproduced from the current one, in contrast to the field theory on ordinary spacetime. We attribute this to the phase-like non-local interaction intrinsic in NC field theory which modifies the analyticity properties of the Green functions significantly.

## 1 Introduction

The idea of describing coordinates in terms of non-commuting operators goes presumably back to Heisenberg and appeared already in a paper by Snyder [1] in 1947, extended by Yang [2] in the same year. It was associated with problems of treating hadrons which are extended objects and the assumption that there might exist a fundamental length. At that time renormalization theory was not yet well developed, but considered as a “distasteful procedure” [1]. Even today this judgment might be shared by some people. In any case the opinion is widespread that at least in a theory which describes consistently all phenomena down to the Planck length the notion of spacetime undergoes some drastic change. The attempts of Connes and Lott [3] aiming at a reformulation of the standard model in terms of non-commutative (NC) geometry started a new era which has been continued in the context of string theory [4]. An analysis which brought the subject neatly into the context of somewhat more conventional quantum field theory has been provided by Doplicher et al. [5]. They showed that there exist representations of the coordinate operators such that for a class of minimal states in the state space of the system interactions can be written as the Moyal product of standard quantum fields as they have been introduced by Filk in another context [6]. The road is then open to perturbation theory which has as a new element just this non-local interaction.

Many calculations and model considerations have been performed in this vein and led to interesting new problems, most noticeably ultraviolet–infrared mixing [7] and a potential violation of unitarity [8]. These were all based on the assumption that up to modifying vertices by NC phase factors the Feynman rules are the usual ones, in particular that lines are represented by the conventional Feynman propagators. Doubts on this have been raised recently in a paper by Bahns et al. [9] by reformulating in  $\varphi^3$  theory the equation of motion in terms of the well-known Yang–Feldman equation [10]. They arrived thereby at a solution which is manifestly Hermitian; hence the theory must be unitary even if time-space NC is non-vanishing. This is, of course, in line with the general considerations of [5], but does not yet explain why one arrives at violation of unitarity in the same model if one uses the Feynman rules of Filk [6], as shown in [8]. Our study shows that the answer is very simple. Starting from standard free field theory, i.e. the standard commutation relations and multi-particle Fock space, one still formally defines the  $S$ -matrix as  $S = T \exp [i \int d^4x \mathcal{L}_{\text{int}}]$ . But when performing the contractions according to Wick’s theorem properly one can never combine the contraction functions of positive and negative frequency to the causal Feynman propagator. This arises because when time does not commute with space, the time-ordering procedure does not commute with the star multiplication either. The naive approach in terms of Feynman propagators is thus not well founded.

In the next section we shall illustrate this statement by a detailed analysis of some examples. The picture on

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how to proceed in the general case will appear naturally in Sect. 3. It turns out that the correct procedure of doing perturbation calculations in NC field theory is just the time-ordered perturbation theory extended to the NC case. We shall spell out the prescriptions for computing  $S$ -matrix and amputated Green functions. We also provide an argument on how the seemingly covariant result in the naive approach cannot be reproduced from the time-ordered perturbation theory. We give our conclusions in the last section.

## 2 Analysis of some examples

The basic quantity in quantum field theory is the Green function which determines the probability amplitude of the  $S$ -matrix. We assume that perturbation theory for NC field theory can still be developed in terms of vacuum expectation values of time-ordered products of field operators so that the Green function is computed as a series expansion as usual,

$$G(x_1, \dots, x_n) = \langle 0 | T \left( \varphi_1(x_1) \cdots \varphi_n(x_n) e^{i \int d^4x \mathcal{L}_{\text{int}}} \right) | 0 \rangle, \quad (1)$$

where  $\mathcal{L}_{\text{int}}$  is the interaction Lagrangian. When one expands the above in  $\mathcal{L}_{\text{int}}$ ,  $T$  is meant to be taken before integration over spacetime is carried out. The time-ordering itself is also defined in the usual manner; for example, for free bosonic fields, it is

$$T(\varphi_1(x_1)\varphi_2(x_2)) = \tau(x_1^0 - x_2^0)\varphi_1(x_1)\varphi_2(x_2) + \tau(x_2^0 - x_1^0)\varphi_2(x_2)\varphi_1(x_1), \quad (2)$$

where  $\tau$  is the step function. (We reserve  $\theta$  for the NC parameter.) In this section we shall work out explicitly some Green functions in NC field theory using the above definitions and free field commutators. The differences to the naive approach of NC field theory will be clearly illustrated by them.

### 2.1 The vertex for the cubic scalar interaction

Let us start with the following three-point function:

$$G(x_1, x_2, x_3) = \int d^4x_4 \langle 0 | T(\varphi(x_1)\varphi(x_2)\varphi(x_3)(\varphi \star \varphi \star \varphi)(x_4)) | 0 \rangle, \quad (3)$$

which would occur in  $\varphi^3$  theory. The star product is defined as

$$(f_1 \star f_2)(x) = \left[ \exp \left( \frac{i}{2} \theta^{\mu\nu} \partial_\mu^x \partial_\nu^y \right) f_1(x) f_2(y) \right]_{y=x}, \quad (4)$$

where  $x, y$  are the usual commutative coordinates and  $\theta_{\mu\nu}$  is a real, antisymmetric, constant matrix characterizing the non-commutativity of spacetime,  $[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$ . As in

the usual field theory, the basic idea to work out the above quantity is to reexpress the time-ordered product in terms of the normal-ordered product ( $N$ ) plus contraction terms arising from field commutators when interchanging field operators. Sandwiched between the vacuum state, only the completely contracted terms will contribute. In the usual field theory this is accomplished by Wick's theorem and a completely contracted term is given by a product of Feynman propagators. The same result is taken for granted in the naive approach of NC field theory. But as we shall show below this is actually not the case when time does not commute with space.

In the example of (3), there are  $4!$  time-orders but we only have to consider two groups of them, namely,  $x_1^0 > x_2^0 > (x_3^0 \text{ and } x_4^0)$  and  $(x_3^0 \text{ and } x_4^0) > x_2^0 > x_1^0$ , while others may be obtained by interchange of coordinate indices. For both of them, the following result will be useful:

$$\begin{aligned} T(\varphi_3\varphi_4 \star \varphi_4 \star \varphi_4) &= N(\varphi_3\varphi_4 \star \varphi_4 \star \varphi_4) \\ &+ N \left( 2D(0)\varphi_3\varphi_4 + \int d^3\mu_{\mathbf{q}} \varphi_3\varphi(x_4 - \tilde{q}) \right) \\ &+ \tau_{34}N(D_{34} \star \varphi_4 \star \varphi_4 + \varphi_4 \star D_{34} \star \varphi_4 + \varphi_4 \star \varphi_4 \star D_{34}) \\ &+ \tau_{43}N(D_{43} \star \varphi_4 \star \varphi_4 + \varphi_4 \star D_{43} \star \varphi_4 + \varphi_4 \star \varphi_4 \star D_{43}) \\ &+ \tau_{34} \left( 2D(0)D_{34} + \int d^3\mu_{\mathbf{q}} D(x_3 - x_4 + \tilde{q}) \right) \\ &+ \tau_{43} \left( 2D(0)D_{43} + \int d^3\mu_{\mathbf{q}} D(x_4 - x_3 - \tilde{q}) \right), \end{aligned} \quad (5)$$

where the star multiplication refers to  $x_4$ . Some explanations are in order. We have denoted the arguments of functions by the indices of their coordinates when no confusion arises; e.g.,  $\varphi_3 = \varphi(x_3)$ ,  $D_{34} = D(x_3 - x_4)$  and  $\tau_{43} = \tau(x_4^0 - x_3^0)$ . The above result is obtained by pushing annihilation (positive-frequency) operators to the right and creation (negative-frequency) operators to the left using the basic commutation relation between them,

$$\begin{aligned} \varphi(x) &= \varphi^+(x) + \varphi^-(x), \\ [\varphi^+(x), \varphi^-(y)] &= D(x - y) \\ &= \int d^3\mu_{\mathbf{p}} \exp[-ip_+ \cdot (x - y)], \end{aligned} \quad (6)$$

where  $d^3\mu_{\mathbf{p}} = d^3\mathbf{p}[(2\pi)^3 2E_{\mathbf{p}}]^{-1}$  is the standard phase space measure with  $E_{\mathbf{p}} = (\mathbf{p}^2 + m^2)^{1/2}$ , and  $p_\lambda^\pm = (\lambda E_{\mathbf{p}}, \mathbf{p})$  ( $\lambda = \pm$ ) is the on-shell momentum with positive or negative energy. Due to NC of the star product we have to take care of the order of ordinary functions in addition to the order of annihilation and creation operators when interchanging operators. For example,

$$\begin{aligned} N(\varphi_4 \star \varphi_4 \star \varphi_4) &= [\varphi^+ \star \varphi^+ \star \varphi^+ + \varphi^- \star \varphi^+ \star \varphi^+ \\ &+ \varphi^- \star \varphi^- \star \varphi^+ + \varphi^- \star \varphi^- \star \varphi^- \\ &+ (\varphi^- \star \varphi^+) \star \varphi^+ + \varphi^- \star (\varphi^+ \star \varphi^+) \\ &+ \varphi^- \star (\varphi^- \star \varphi^+) + (\varphi^- \star \varphi^-) \star \varphi^+] (x_4) \\ &+ 2D(0)\varphi_4 + \int d^3\mu_{\mathbf{q}} \varphi(x_4 - \tilde{q}), \end{aligned} \quad (7)$$

where  $\tilde{q}_\mu = \theta_{\mu\nu} q^\nu$  and  $\star$  refers to the star product using  $-\theta_{\mu\nu}$ . The last but one term arises from contractions of the

left and right fields with the middle one which is divergent as usual, while the last term comes from the contraction of the left with the right which is non-local because of the star product. These two terms are the origin of the third line and the last two lines in (5). These terms will finally contribute to the disconnected part of  $G(x_1, x_2, x_3)$  and are thus ignored from now on. The second line in (5) does not contribute either when multiplied from the left by  $\varphi_1\varphi_2$  or from the right by  $\varphi_2\varphi_1$  and sandwiched between the vacuum. We are thus left with the fourth and fifth lines in (5). Before we proceed, we would like to stress that for NC time-space (i.e.,  $\theta_{0i} \neq 0$ ) it is not permitted to change the order of time-ordering and the star multiplication since the former involves a distribution functions of time and the latter contains derivatives with respect to time. In this general case, we cannot move  $\tau_{34}$  and  $\tau_{43}$  inside of the star product to form with  $D_{34}$  and  $D_{43}$  the Feynman propagator  $D_F(x_3 - x_4)$ . This is the main source of the difference in the current approach from the conventional one pursued in the literature. Only when time commutes with space, the two approaches become identical.

Let us go back to the computation of  $G(x_1, x_2, x_3)$  and consider the time sequence  $x_1^0 > x_2^0 > x_3^0 > x_4^0$ ,

$$A = \langle 0 | \varphi_1 \varphi_2 N(D_{34} \star \varphi_4 \star \varphi_4 + \varphi_4 \star D_{34} \star \varphi_4 + \varphi_4 \star \varphi_4 \star D_{34}) | 0 \rangle. \quad (8)$$

Up to disconnected terms, only the combination

$$\varphi_1^+ \varphi_2^+ \cdots \varphi_4^- \cdots \varphi_4^- \cdots$$

contributes. Applying repeatedly

$$\varphi_1^+ \varphi_2^+ \varphi_4^- = \varphi_4^- \varphi_1^+ \varphi_2^+ + D_{24} \varphi_1^+ + D_{14} \varphi_2^+, \quad (9)$$

we obtain

$$A = \{D_{14}, D_{24}, D_{34}\}_\star, \quad (10)$$

where the star refers only to  $x_4$  and we have introduced the completely symmetrized sum of the star products,

$$\{B_1, B_2, B_3\}_\star = \sum_{\pi_3} B_{\pi(1)} \star B_{\pi(2)} \star B_{\pi(3)}, \quad (11)$$

where  $\pi_3$  is the permutation of three objects. The above result is symmetric in  $x_1$  and  $x_2$  and thus applies as well to the time sequence  $x_2^0 > x_1^0 > x_3^0 > x_4^0$ . Including the case of  $x_4^0 > x_3^0$ , we have, for  $T_{12}T_{34}$  and up to disconnected terms,

$$\begin{aligned} & \langle 0 | T(\varphi_1 \varphi_2 \varphi_3 (\varphi \star \varphi \star \varphi)(x_4)) | 0 \rangle \\ &= \tau_{34} \{D_{14}, D_{24}, D_{34}\}_\star + \tau_{43} \{D_{14}, D_{24}, D_{43}\}_\star, \end{aligned} \quad (12)$$

where  $T_{12}T_{34}$  denotes the time sequence  $(x_1^0 \text{ and } x_2^0) > (x_3^0 \text{ and } x_4^0)$ . The opposite case of  $(x_3^0 \text{ and } x_4^0) > (x_1^0 \text{ and } x_2^0)$  is similarly computed to be

$$T_{34}T_{12} : \tau_{34} \{D_{41}, D_{42}, D_{34}\}_\star + \tau_{43} \{D_{41}, D_{42}, D_{43}\}_\star. \quad (13)$$

The other two pairs of cases are obtained by permutation of the indices 1, 2, 3:

$$\begin{aligned} T_{23}T_{14} : & \tau_{14} \{D_{24}, D_{34}, D_{14}\}_\star + \tau_{41} \{D_{24}, D_{34}, D_{41}\}_\star, \\ T_{14}T_{23} : & \tau_{14} \{D_{42}, D_{43}, D_{14}\}_\star + \tau_{41} \{D_{42}, D_{43}, D_{41}\}_\star, \\ T_{31}T_{24} : & \tau_{24} \{D_{34}, D_{14}, D_{24}\}_\star + \tau_{42} \{D_{34}, D_{14}, D_{42}\}_\star, \\ T_{24}T_{31} : & \tau_{24} \{D_{43}, D_{41}, D_{24}\}_\star + \tau_{42} \{D_{43}, D_{41}, D_{42}\}_\star. \end{aligned} \quad (14)$$

The above results can be combined into a compact expression. Since the connected contribution involves exclusively functions  $D_{j4}$  and  $D_{4j}$  ( $j = 1, 2, 3$ ), we hope that they are also accompanied exclusively by step functions  $\tau_{j4}$  and  $\tau_{4j}$ . This is indeed the case as in ordinary field theory. Actually all terms except  $\{D_{14}, D_{24}, D_{34}\}_\star$  and  $\{D_{41}, D_{42}, D_{43}\}_\star$  are already in the desired form. There are three contributions to  $\{D_{14}, D_{24}, D_{34}\}_\star$  arising respectively from the time sequences  $(x_1^0 \text{ and } x_2^0) > x_3^0 > x_4^0$ ,  $(x_2^0 \text{ and } x_3^0) > x_1^0 > x_4^0$  and  $(x_3^0 \text{ and } x_1^0) > x_2^0 > x_4^0$ , whose union is identical to the sequence  $(x_1^0 \text{ and } x_2^0 \text{ and } x_3^0) > x_4^0$  corresponding to the product of step functions  $\tau_{14}\tau_{24}\tau_{34}$ . A similar combination occurs for  $\{D_{41}, D_{42}, D_{43}\}_\star$ , so that

$$\begin{aligned} & \langle 0 | T(\varphi_1 \varphi_2 \varphi_3 (\varphi \star \varphi \star \varphi)(x_4)) | 0 \rangle \\ &= \sum_{\lambda_1 \lambda_2 \lambda_3} \tau_{14}^{\lambda_1} \tau_{24}^{\lambda_2} \tau_{34}^{\lambda_3} \{D_{14}^{\lambda_1}, D_{24}^{\lambda_2}, D_{34}^{\lambda_3}\}_\star, \end{aligned} \quad (15)$$

where  $\lambda_j = \pm$  defines the direction of time in the relevant function,

$$\tau_{j4}^{\lambda_j} = \begin{cases} \tau_{j4}, & D_{j4}^{\lambda_j} = \begin{cases} D_{j4}, & \text{for } \lambda_j = +, \\ D_{4j}, & \text{for } \lambda_j = -. \end{cases} \end{cases} \quad (16)$$

We would like to emphasize again that it is generally not permitted to move the step functions into the star products in (15). Only when time commutes with space, we are allowed to do so and the result of the naive approach is reproduced,

$$\{D_F(x_1 - x_4), D_F(x_2 - x_4), D_F(x_3 - x_4)\}_\star, \quad \text{for } \theta_{0i} = 0, \quad (17)$$

where  $D_F(x_j - x_4) = \tau_{j4}^+ D_{j4}^+ + \tau_{j4}^- D_{j4}^-$  is the Feynman propagator in coordinate space.

To transform into momentum space we first work out explicitly the quantity in (15). Using the expressions

$$\begin{aligned} \tau_{j4}^{\lambda_j} &= \frac{i\lambda_j}{2\pi} \int_{-\infty}^{\infty} ds_j \frac{\exp[-is_j(x_j^0 - x_4^0)]}{s_j + i\epsilon\lambda_j}, \\ D_{j4}^{\lambda_j} &= \int d^3\mu_{\mathbf{p}_j} \exp[-ip_j\lambda_j \cdot (x_j - x_4)], \end{aligned} \quad (18)$$

we have, for example,

$$\begin{aligned} & \tau_{14}^{\lambda_1} \tau_{24}^{\lambda_2} \tau_{34}^{\lambda_3} \left( D_{14}^{\lambda_1} \star D_{24}^{\lambda_2} \star D_{34}^{\lambda_3} \right) \\ &= \prod_{j=1}^3 \left[ \frac{i\lambda_j}{2\pi} \int_{-\infty}^{\infty} ds_j \int d^3\mu_{\mathbf{p}_j} \frac{e^{-ip_j \cdot (x_j - x_4)}}{s_j + i\epsilon\lambda_j} \right] \end{aligned}$$

$$\begin{aligned}
& \times \exp[-i(p_{1\lambda_1}, p_{2\lambda_2}, p_{3\lambda_3})] \\
& = \prod_{j=1}^3 \left[ \int \frac{d^4 p_j}{(2\pi)^4} i P_{\lambda_j}(p_j) e^{-ip_j \cdot (x_j - x_4)} \right] \\
& \times \exp[-i(p_{1\lambda_1}, p_{2\lambda_2}, p_{3\lambda_3})], \tag{19}
\end{aligned}$$

where we introduced  $p_j^0 = s_j + \lambda_j E_{\mathbf{p}_j}$  and

$$\begin{aligned}
P_\lambda(k) & = \frac{\lambda}{2E_{\mathbf{k}}[k^0 - \lambda(E_{\mathbf{k}} - i\epsilon)]} \\
& = (2E_{\mathbf{k}}[\lambda k^0 - (E_{\mathbf{k}} - i\epsilon)])^{-1}, \\
(k_1, k_2, \dots, k_n) & = \sum_{i < j} k_i \wedge k_j, \tag{20}
\end{aligned}$$

with  $p \wedge q = 1/2 \theta_{\mu\nu} p^\mu q^\nu$ . The pair of parentheses introduced above has some properties to be used implicitly later on. It changes sign when the order of arguments is reversed. The shift of sign in some arguments is identical to the shift of sign in the remaining arguments. It is important to note that the wedge product in the NC phase of (19) involves only on-shell momenta of positive ( $\lambda_j = +$ ) or negative ( $\lambda_j = -$ ) energy corresponding to propagation in the time direction of  $x_j^0 > x_4^0$  or  $x_j^0 < x_4^0$ . Including all permutations of  $D$  factors in (15) which amounts to summing over permutations of the NC phase, and integrating over  $x_4$ , we arrive at

$$\begin{aligned}
& G(x_1, x_2, x_3) \\
& = \sum_{\lambda_1 \lambda_2 \lambda_3} \prod_{j=1}^3 \left[ \int \frac{d^4 p_j}{(2\pi)^4} i P_{\lambda_j}(p_j) \exp(-ip_j \cdot x_j) \right] \\
& \times (2\pi)^4 \delta^4(p_1 + p_2 + p_3) \tag{21} \\
& \times \sum_{\pi_3} \exp[-i(p_{\pi(1)\lambda_{\pi(1)}}, p_{\pi(2)\lambda_{\pi(2)}}, p_{\pi(3)\lambda_{\pi(3)}})].
\end{aligned}$$

Transforming into momentum space is now straightforward:

$$\begin{aligned}
\hat{G}(k_1, k_2, k_3) & = \prod_{j=1}^3 \left[ \int d^4 x_j \exp(-ik_j \cdot x_j) \right] \\
& \times G(x_1, x_2, x_3) \\
& = (2\pi)^4 \delta^4(k_1 + k_2 + k_3) \sum_{\lambda_1 \lambda_2 \lambda_3} \prod_{j=1}^3 [i P_{\lambda_j}(k_j)] \\
& \times \sum_{\pi_3} \exp[-i(k_{\pi(1)\lambda_{\pi(1)}}, k_{\pi(2)\lambda_{\pi(2)}}, k_{\pi(3)\lambda_{\pi(3)}})], \tag{22}
\end{aligned}$$

where  $k_j$ 's are the incoming momenta into the vertex. We have reversed the signs of variables  $\lambda_j$  and  $\lambda$ , used  $P_{-\lambda}(-k) = P_\lambda(k)$  and the property of the parentheses to remove the minus signs in the arguments of the NC phases.

We make a few comments on the above result. First, the NC phases involve only on-shell momenta of positive and negative energies. For a given set of  $\lambda_j$ , the permutation sum of NC phases is

$$\begin{aligned}
& 2 \cos(k_{1\lambda_1}, k_{2\lambda_2}, k_{3\lambda_3}) + 2 \cos(k_{2\lambda_2}, k_{3\lambda_3}, k_{1\lambda_1}) \\
& + 2 \cos(k_{3\lambda_3}, k_{1\lambda_1}, k_{2\lambda_2}). \tag{23}
\end{aligned}$$

Note that the above does not simplify into  $6 \cos(k_{1\lambda_1}, k_{2\lambda_2})$  etc. as it does in the naive approach where all  $k_j \lambda_j$  are replaced by  $k_j$ . The reason is that while  $\sum_j k_j = 0$  is always true this is generally not the case with  $k_j \lambda_j$  where  $k_j^0$  is replaced by  $\lambda_j E_{\mathbf{k}_j}$ : even if this sum vanishes for some configuration of  $\lambda_j$  and  $\mathbf{k}_j$ , it cannot vanish for all configurations. For the case of identical fields considered here, there is even no such configuration at all due to kinematics. Furthermore, since the NC phases depend on the time direction parameters  $\lambda_j$  we cannot exhaust the sum over  $\lambda_j$  by using  $iP_+(k) + iP_-(k) = i\hat{D}_F(k)$  with  $i\hat{D}_F(k) = i(k^2 - m^2 + i\epsilon)^{-1}$  being the Feynman propagator. These findings are completely different from the naive approach. It is intriguing that such differences occur already at tree level in perturbation theory and we thus expect that the whole picture of perturbation theory will be altered. The differences arise from the fact that we are in general not allowed to interchange the order of the time-ordering procedure and the star multiplication. Only when  $\theta_{0i} = 0$ , the star multiplication does not involve time derivatives and the NC phases in (22) are independent of  $\lambda_j$ , and then the differences disappear.

The external lines in  $\hat{G}$  may be amputated by multiplying by an inverse Feynman propagator  $(i\hat{D}_F)^{-1}$  for each external line and noting that

$$\begin{aligned}
P_\lambda(k) & = \hat{D}_F(k) \eta_\lambda(k), \\
\eta_\lambda(k) & = \frac{1}{2} \left( 1 + \lambda \frac{k_0}{E_{\mathbf{k}}} \right). \tag{24}
\end{aligned}$$

We obtain the 1PI vertex for the above  $\hat{G}$ :

$$\begin{aligned}
& \hat{\Gamma}(k_1, k_2, k_3) \\
& = (2\pi)^4 \delta^4 \left( \sum_i k_i \right) \sum_{\{\lambda_j\}} \prod_{j=1}^3 [\eta_{\lambda_j}(k_j)] \\
& \times \sum_{\pi_3} \exp[-i(k_{\pi(1)\lambda_{\pi(1)}}, k_{\pi(2)\lambda_{\pi(2)}}, k_{\pi(3)\lambda_{\pi(3)}})]. \tag{25}
\end{aligned}$$

## 2.2 The two by two scattering through cubic interactions

To motivate our generalization in the next section, we consider the following four-point function,

$$\begin{aligned}
& G(x_1, x_2, x_3, x_4) \\
& = \int d^4 x_5 \int d^4 x_6 \\
& \times \langle 0 | T (\pi_1 \pi_2 \chi_3 \chi_4 (\pi \star \sigma \star \pi)_5 (\chi \star \sigma \star \chi)_6) | 0 \rangle, \tag{26}
\end{aligned}$$

which would arise from the Lagrangian of cubic interactions amongst real scalar fields,

$$\mathcal{L}_{\text{int}} = -g_\pi (\pi \star \sigma \star \pi)(x) - g_\chi (\chi \star \sigma \star \chi)(x). \tag{27}$$

We have deliberately introduced non-identical fields to avoid unnecessary complications due to many possible

contractions amongst factors of identical fields, which just amounts to proper symmetrization of the NC phases, as we saw in the above example. Our goal will be the  $S$ -matrix element of the two by two scattering  $\pi\pi \rightarrow \chi\chi$  and its crossed channels.

It is clear that we should first contract the two  $\sigma$  fields. We have, for  $x_5^0 > x_6^0$ ,

$$\begin{aligned} & \langle 0|T(\pi_1\pi_2\chi_3\chi_4(\pi\star\sigma\star\pi)_5(\chi\star\sigma\star\chi)_6)|0\rangle \\ &= \int d^3\mu_{\mathbf{p}} \langle 0|\cdots(\pi\star e^{-ip_+\cdot x_5}\star\pi)_5\cdots \\ & \quad \times(\chi\star e^{+ip_+\cdot x_6}\star\chi)_6\cdots|0\rangle, \end{aligned} \quad (28)$$

and for  $x_5^0 < x_6^0$ ,

$$\begin{aligned} & \langle 0|T(\pi_1\pi_2\chi_3\chi_4(\pi\star\sigma\star\pi)_5(\chi\star\sigma\star\chi)_6)|0\rangle \\ &= \int d^3\mu_{\mathbf{p}} \langle 0|\cdots(\chi\star e^{-ip_+\cdot x_6}\star\chi)_6\cdots \\ & \quad \times(\pi\star e^{+ip_+\cdot x_5}\star\pi)_5\cdots|0\rangle, \end{aligned} \quad (29)$$

where the dots represent other fields appropriate to the time-ordering and  $p$  refers to the  $\sigma$  field, especially  $p_{\pm}^0 = \pm E_{\mathbf{p}} = \pm(\mathbf{p}^2 + m_{\sigma}^2)^{1/2}$ .

Next we consider contractions of  $\pi$  and  $\chi$  fields. Since  $[\pi_i, \chi_j] = 0$ , the relative order of  $\pi$  and  $\chi$  fields is irrelevant; what is relevant is the order within the groups  $(x_1^0, x_2^0, x_5^0)$  and  $(x_3^0, x_4^0, x_6^0)$  respectively. Corresponding to  $x_5^0 > x_6^0$  and  $x_6^0 > x_5^0$ , we have two possibilities,  $T_{125}T_{346}$  and  $T_{346}T_{125}$ . Let us study the  $\pi$  field contraction,

$$\langle 0|\cdots T(\pi_1\pi_2\pi_5\star e^{\pm ip_+\cdot x_5}\star\pi_5)\cdots|0\rangle. \quad (30)$$

There are 3! orders. For example, for  $x_1^0 > x_2^0 > x_5^0$ , the above becomes, up to disconnected terms,

$$\begin{aligned} & \langle 0|\cdots\pi_1^+\pi_2^+\pi_5^-\star e^{\pm ip_+\cdot x_5}\star\pi_5^-\cdots|0\rangle \\ &= \langle 0|\cdots(D_{15}\pi_2^+ + D_{25}\pi_1^+)\star e^{\pm ip_+\cdot x_5}\star\pi_5^-\cdots|0\rangle \\ &= (D_{15}\star e^{\pm ip_+\cdot x_5}\star D_{25} + (1\leftrightarrow 2))\langle 0|\cdots|0\rangle, \end{aligned} \quad (31)$$

where  $D_{15}$  and  $D_{25}$  refer to the  $\pi$  field and  $\star$  refers to  $x_5$ . The above is symmetric in  $x_1$  and  $x_2$  and thus actually corresponds to the time-order specified by  $\tau_{15}\tau_{25}$ . The other time-orders can be similarly computed. Their sum gives the complete result for all orders,

$$\begin{aligned} & \langle 0|\cdots T(\pi_1\pi_2\pi_5\star e^{\pm ip_+\cdot x_5}\star\pi_5)\cdots|0\rangle \\ &= \sum_{\lambda_1\lambda_2} \tau_{15}^{\lambda_1}\tau_{25}^{\lambda_2} (D_{15}^{\lambda_1}\star e^{\pm ip_+\cdot x_5}\star D_{25}^{\lambda_2} + (1\leftrightarrow 2))\langle 0|\cdots|0\rangle. \end{aligned} \quad (32)$$

The last factor in the above is precisely the one for the  $\chi$  field contraction and is similarly computed. We thus have

$$\begin{aligned} & \langle 0|T(\pi_1\pi_2\chi_3\chi_4(\pi\star\sigma\star\pi)_5(\chi\star\sigma\star\chi)_6)|0\rangle \\ &= \int d^3\mu_{\mathbf{p}} \tau_{56} \sum_{\{\lambda_j\}} \tau_{15}^{\lambda_1}\tau_{25}^{\lambda_2}\tau_{36}^{\lambda_3}\tau_{46}^{\lambda_4} \\ & \quad \times \left[ D_{15}^{\lambda_1}\star e^{-ip_+\cdot x_5}\star D_{25}^{\lambda_2} + (1\leftrightarrow 2) \right] \\ & \quad \times \left[ D_{36}^{\lambda_3}\star e^{+ip_+\cdot x_6}\star D_{46}^{\lambda_4} + (3\leftrightarrow 4) \right] \end{aligned}$$

$$\begin{aligned} & + (\text{same as above except } \tau_{56} \rightarrow \tau_{65}, x_{5,6} \rightarrow -x_{5,6}) \\ &= \int d^3\mu_{\mathbf{p}} \sum_{\{\lambda_j\}} \sum_{\lambda} \tau_{15}^{\lambda_1}\tau_{25}^{\lambda_2}\tau_{36}^{\lambda_3}\tau_{46}^{\lambda_4}\tau_{56}^{\lambda} \\ & \quad \times \left[ D_{15}^{\lambda_1}\star e^{-ip_{\lambda}\cdot x_5}\star D_{25}^{\lambda_2} + (1\leftrightarrow 2) \right] \\ & \quad \times \left[ D_{36}^{\lambda_3}\star e^{+ip_{\lambda}\cdot x_6}\star D_{46}^{\lambda_4} + (3\leftrightarrow 4) \right], \end{aligned} \quad (33)$$

where  $D_{36}^{\lambda_3}$  and  $D_{46}^{\lambda_4}$  refer to the  $\chi$  field and the star in the second factor is with respect to  $x_6$ . In the second equality we have made the shift  $\mathbf{p} \rightarrow -\mathbf{p}$  for  $\lambda = -$ .

Using the representations as shown in (18) and the same trick that led to (19), we make the above Green function ready for transformation into momentum space. For example,

$$\begin{aligned} & \tau_{15}^{\lambda_1}\tau_{25}^{\lambda_2}\tau_{36}^{\lambda_3}\tau_{46}^{\lambda_4}\tau_{56}^{\lambda} \\ & \quad \times \left[ D_{15}^{\lambda_1}\star e^{-ip_{\lambda}\cdot x_5}\star D_{25}^{\lambda_2} \right] \left[ D_{36}^{\lambda_3}\star e^{+ip_{\lambda}\cdot x_6}\star D_{46}^{\lambda_4} \right] \\ &= \prod_{j=1}^4 \left[ \int \frac{d^4p_j}{(2\pi)^4} iP_{\lambda_j}(p_j) \right] \int \frac{d^4p}{(2\pi)^4} iP_{\lambda}(p) \\ & \quad \times e^{-ip_1\cdot(x_1-x_5)} e^{-ip_2\cdot(x_2-x_5)} e^{-ip_3\cdot(x_3-x_6)} \\ & \quad \times e^{-ip_4\cdot(x_4-x_6)} e^{-ip\cdot(x_5-x_6)} \\ & \quad \times \exp[-i(p_{1\lambda_1}, -p_{\lambda}, p_{2\lambda_2})] \exp[-i(p_{3\lambda_3}, +p_{\lambda}, p_{4\lambda_4})]. \end{aligned} \quad (34)$$

We can now integrate over  $x_5$  and  $x_6$ , which results in two factors of  $\delta$  functions, then transform into momentum space, and sum over all terms,

$$\begin{aligned} & \hat{G}(k_1, k_2, k_3, k_4) \\ &= \prod_{j=1}^4 \left[ \int d^4x_j e^{-ik_j\cdot x_j} \right] G(x_1, x_2, x_3, x_4) \\ &= (2\pi)^4 \delta^4 \left( \sum_i k_i \right) \sum_{\{\lambda_j\}} \sum_{\lambda} \prod_j [iP_{\lambda_j}(k_j)] iP_{\lambda}(p) \\ & \quad \times (\exp[-i(k_{1\lambda_1}, -p_{\lambda}, k_{2\lambda_2})] + (1\leftrightarrow 2)) \\ & \quad \times (\exp[-i(k_{3\lambda_3}, +p_{\lambda}, k_{4\lambda_4})] + (3\leftrightarrow 4)) \\ &= (2\pi)^4 \delta^4 \left( \sum_i k_i \right) \sum_{\{\lambda_j\}} \sum_{\lambda} \prod_j [iP_{\lambda_j}(k_j)] iP_{\lambda}(p) \\ & \quad \times 2 \cos(k_{1\lambda_1}, -p_{\lambda}, k_{2\lambda_2}) 2 \cos(k_{3\lambda_3}, +p_{\lambda}, k_{4\lambda_4}), \end{aligned} \quad (35)$$

with  $p = k_1 + k_2 = -k_3 - k_4$ . Be careful that  $k_{1\lambda_1} + k_{2\lambda_2} \neq p_{\lambda} \neq -k_{3\lambda_3} - k_{4\lambda_4}$ . For comparison, in the naive approach the above would be

$$\begin{aligned} & (2\pi)^4 \delta^4 \left( \sum_i k_i \right) \prod_j [i\hat{D}_F(k_j)] i\hat{D}_F(p) \\ & \quad \times 2 \cos(k_1, k_2) 2 \cos(k_3, k_4), \end{aligned} \quad (36)$$

which according to our preceding analysis is correct only for  $\theta_{0i} = 0$ . The 1PI function is obtained by amputation,

$$\hat{\Gamma}(k_1, k_2, k_3, k_4)$$

$$\begin{aligned}
&= (2\pi)^4 \delta^4 \left( \sum_i k_i \right) \sum_{\{\lambda_j\}} \sum_{\lambda} \prod_j [\eta_{\lambda_j}(k_j)] iP_{\lambda}(p) \\
&\times 2 \cos(k_{1\lambda_1}, -p_{\lambda}, k_{2\lambda_2}) 2 \cos(k_{3\lambda_3}, +p_{\lambda}, k_{4\lambda_4}). \quad (37)
\end{aligned}$$

Let us now extract the  $S$ -matrix element for on-shell particles from the above 1PI function. We take the example of  $\pi\pi \rightarrow \chi\chi$  scattering. This means that  $k_1^0 = +E_{\mathbf{k}_1}$ ,  $k_2^0 = +E_{\mathbf{k}_2}$ ,  $k_3^0 = -E_{\mathbf{k}_3}$ ,  $k_4^0 = -E_{\mathbf{k}_4}$ . Thus only one term in the sum over  $\{\lambda_j\}$  contributes due to  $\eta_-(k_1) = 0$  etc. Including the coupling factors as well, the transition amplitude is

$$\begin{aligned}
&iT(\pi(k_1) + \pi(k_2) \rightarrow \chi(k_3) + \chi(k_4)) \\
&= (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - k_4) \sum_{\lambda} iP_{\lambda}(p) \\
&\times (-ig_{\pi}) 2 \cos(k_{1+}, -p_{\lambda}, k_{2+}) \\
&\times (-ig_{\chi}) 2 \cos(-k_{3+}, p_{\lambda}, -k_{4+}) \\
&= (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - k_4) i\mathcal{A}, \quad (38)
\end{aligned}$$

where  $p = k_1 + k_2 = k_3 + k_4$  and  $\mathcal{A}$  is the amplitude with the usual normalization as computed from Feynman diagrams in ordinary quantum field theory. Note also that we have reversed the signs of  $k_3$  and  $k_4$  so that  $k_{3-} \rightarrow -k_{3+}$ ,  $k_{4-} \rightarrow -k_{4+}$ . More explicitly,

$$\begin{aligned}
&T(\pi(k_1) + \pi(k_2) \rightarrow \chi(k_3) + \chi(k_4)) \\
&= -g_{\pi} g_{\chi} (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - k_4) \\
&\times \left[ \frac{2 \cos(k_{1+}, -p_+, k_{2+}) 2 \cos(-k_{3+}, p_+, -k_{4+})}{2E_{\mathbf{k}_1 + \mathbf{k}_1} (k_1^0 + k_2^0 - E_{\mathbf{k}_1 + \mathbf{k}_1} + i\epsilon)} \right. \\
&\left. + \frac{2 \cos(-k_{3+}, p_-, -k_{4+}) 2 \cos(k_{1+}, -p_-, k_{2+})}{2E_{\mathbf{k}_3 + \mathbf{k}_4} (-k_3^0 - k_4^0 - E_{\mathbf{k}_3 + \mathbf{k}_4} + i\epsilon)} \right]. \quad (39)
\end{aligned}$$

We make a few remarks concerning the  $S$ -matrix calculation. First, the crossed channels of the above process may be obtained similarly. For example, for  $\pi\chi \rightarrow \pi\chi$  scattering, we may choose  $k_1^0 = +E_{\mathbf{k}_1}$ ,  $k_2^0 = -E_{\mathbf{k}_2}$ ,  $k_3^0 = +E_{\mathbf{k}_3}$ ,  $k_4^0 = -E_{\mathbf{k}_4}$ . Second, from the above detailed analysis it is clear how to calculate the most complicated case of identical particle scattering through their self-interactions. We should include all possible Feynman diagrams and for each of them employ the same analysis which just amounts to more symmetrization at the vertices with respect to identical fields. In this way we get the following contributions to the amputated four-point Green function of the  $\pi$  field at the lowest level in  $\mathcal{L}_{\text{int}} = -g_{\pi} \pi \star \pi \star \pi$ ,

$$\begin{aligned}
&\hat{I}(k_1, k_2, k_3, k_4) \\
&= -g_{\pi}^2 (2\pi)^4 \delta^4 \left( \sum_i k_i \right) \sum_{\{\lambda_j\}} \sum_{\lambda} \prod_{j=1}^4 [\eta_{\lambda_j}(k_j)] \\
&\times (A_s + A_t + A_u), \quad (40)
\end{aligned}$$

where  $A_s, A_t, A_u$  are from  $s-, t-, u-$ channels respectively,

$$A_s = iP_{\lambda}(p_s) \sum_{\pi_3} \exp[-i(k_{1\lambda_1}, k_{2\lambda_2}, -p_{s\lambda})]$$

$$\times \sum_{\pi_3} \exp[-i(k_{3\lambda_3}, k_{4\lambda_4}, +p_{s\lambda})],$$

$$A_t = iP_{\lambda}(p_t) \sum_{\pi_3} \exp[-i(k_{1\lambda_1}, k_{3\lambda_3}, -p_{t\lambda})]$$

$$\times \sum_{\pi_3} \exp[-i(k_{2\lambda_2}, k_{4\lambda_4}, +p_{t\lambda})],$$

$$A_u = iP_{\lambda}(p_u) \sum_{\pi_3} \exp[-i(k_{1\lambda_1}, k_{4\lambda_4}, -p_{u\lambda})]$$

$$\times \sum_{\pi_3} \exp[-i(k_{3\lambda_3}, k_{2\lambda_2}, +p_{u\lambda})], \quad (41)$$

with  $p_s = k_1 + k_2$ ,  $p_t = k_1 + k_3$ ,  $p_u = k_1 + k_4$ . Here  $\pi_3$  refers to the  $3!$  permutations of the three momenta appearing in each factor of the NC phase sums. It is straightforward to project the  $S$ -matrix element from the above which we shall not write down. And there is also no problem to extend to more complicated interactions like  $\varphi^4$ . Finally, if our aim is restricted to the  $S$ -matrix for on-shell particles, we may proceed more directly from the expectation values of the  $S$ -operator. For the above example, we need compute the following quantity,

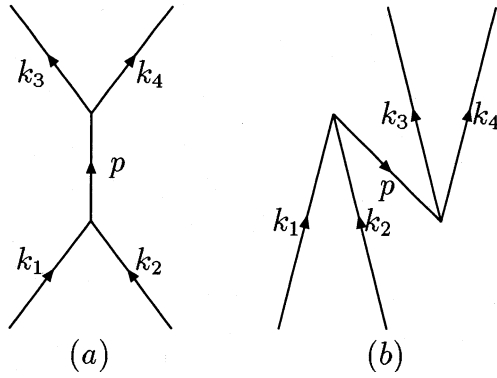
$$\begin{aligned}
\langle \chi\chi | S | \pi\pi \rangle &= (-ig_{\pi})(-ig_{\chi}) \int d^4x_5 \int d^4x_6 (\chi(k_{3+})\chi(k_{4+}) | \\
&\times T((\pi \star \sigma \star \pi)_5 (\chi \star \sigma \star \chi)_6) | \pi(k_{1+})\pi(k_{2+}) \rangle \\
&+ \text{higher orders}, \quad (42)
\end{aligned}$$

which just corresponds to a special assignment of the time-order in (26), namely  $\pi_1$  and  $\pi_2$  in the far past,  $\chi_3$  and  $\chi_4$  in the far future, and others in between. This is precisely the contributing part in the above analysis. There is thus no doubt that the results for the  $S$ -matrix coincide. The main advantage of coping with the Green function for this purpose is that we may project all physical processes from the same Green function.

### 3 Generalization

The structure shown in (39) looks familiar to us and is very suggestive. Actually it is nothing but the ‘‘old-fashioned’’, time-ordered perturbation theory [11,12] properly modified to NC field theory. This fits also on a somewhat more formal level to the pragmatic point of view which we assume here. The  $S$ -operator maps on the one hand any prepared incoming state onto the respective outgoing one, but this can also be considered as the time transport of this incoming state into the outgoing one. As long as we can represent it as a time-ordered exponential with a Hermitian exponent times the pure imaginary unit  $i$  we have formal unitarity and thus satisfy the first requirement for true unitarity which means conservation of the transition probability in the sense of quantum mechanics – to which we shall return in a separate publication.

The time-ordered Feynman diagrams for the above example are depicted in Fig. 1. In the language of the time-ordered perturbation theory, a physical process is virtualized as a series of transitions between physical intermedi-



**Fig. 1.** Time-ordered diagrams corresponding to (39). Time flows upwards

ate states that are sequential in time. The transition amplitude is weighted by the interaction vertices which are evaluated for on-shell momenta if they depend on them and by the energy deficit of the intermediate states. Realizing this, it becomes obvious how to proceed in the general case. For further applications, we give below the prescriptions for computing the on-shell transition matrix  $T$  at some fixed order in perturbation, which are readily generalized from the ordinary ones [11, 12]. The additional piece for general off-shell amputated Green functions will be described later on.

(1) Draw all Feynman diagrams for the process under consideration. For each Feynman diagram draw all of its time-ordered diagrams. Only the time-order of interacting vertices is relevant and indistinguishable time-orders are counted only once. Each time-ordered diagram is computed by putting together the following factors.

(2) Associate with each internal line (with a spatial momentum  $\mathbf{k}$ ) a phase space integral  $\int d^3\mu_{\mathbf{k}}$ .

(3) Associate with each vertex  $v$ , which is formed by internal lines  $j$  and external ones  $e$  of incoming spatial momenta  $\mathbf{q}_a$  and which has the interaction pattern in the Lagrangian  $-g\psi_1 \star \psi_2 \star \dots \star \psi_n$ , an interaction factor  $g \exp[-i(q_{1\lambda_1}, q_{2\lambda_2}, \dots, q_{n\lambda_n})]$ .  $\lambda_a = +(-)$  if the vertex  $v$  is the later (earlier) end of the line  $a$ . The initial (final) particles  $e$  are always counted as earlier (later) than the vertex  $v$ . Symmetrize the above factor with respect to identical fields. Impose spatial momentum conservation at the vertex  $v$  by multiplying  $(2\pi)^3 \delta^3(\sum_a \mathbf{q}_a)$ .

(4) Associate with each intermediate state occurring between two sequential vertices (earlier  $v_1$  and later  $v_2$ ) a factor of energy deficit,  $[\sum_e (\pm p_e^0) - \sum_j E(\mathbf{k}_j) + i\epsilon]^{-1}$ . Here  $\sum_e (\pm p_e^0)$  is the algebraic sum of the zeroth components of external momenta entering (+) or leaving (-) the diagram before and including the earlier vertex  $v_1$ .  $E(\mathbf{k}_j)$  is the on-shell positive energy of the  $j$ th line contained in this intermediate state.

(5) Multiply by a global factor of  $-2\pi\delta(\sum_e (\pm p_e^0))$  and a symmetry factor  $1/S$  which excludes that of indistinguishable diagrams mentioned above.

The above prescriptions would be precisely the same as obtained in ordinary field theory if we could interpret

the vertex factor as a kind of numerator arising from spin. In ordinary relativistic field theory we can recast the time-ordered perturbation theory into a covariant form in terms of Feynman diagrams. So, it is tempting to ask why this is not possible in NC field theory. Of course, Lorentz invariance is lost at the very beginning and it is not guaranteed that a seemingly covariant formalism exists and is equivalent to the time-ordered one if it does. But this is not the whole point. As far as Feynman diagrams are concerned, we can always treat  $\theta_{\mu\nu}$  as if it were a Lorentz tensor and there will be no problem if we do not use any special reference frame for the calculation since we could not return back by a transformation afterwards [13]. We could also consider  $\theta_{\mu\nu}$  as some background field and assign to it a transformation law so that the above consequence still applies. In the following we present an argument that in NC field theory formulated via time-ordered perturbation theory one cannot reproduce the seemingly covariant results of the naive approach. Our time-ordered version seems however to be a safe starting point as far as quantum mechanics still applies to NC spacetime. A key element of it is the highly non-local character of NC interactions.

Let us first recall briefly how to shift from the covariant perturbation theory to the time-ordered one in ordinary field theory. For a detailed account of the topic we refer the interested reader to [12] for a nice presentation. For this purpose, we consider the contribution from a Feynman diagram to a general, unamputated and connected Green function in momentum space. One first expresses the  $\delta$  function of the zeroth components of momenta at each vertex in terms of a time integral. Then, one rearranges the product of time integrals thus obtained in a time-ordered way. This is followed by integrating over the zeroth momentum components, which is typically of the following form:

$$\int_{-\infty}^{\infty} dk_0 \frac{ie^{-ik_0 t} f(k_0)}{k^2 - m^2 + i\epsilon} = \int_{-\infty}^{\infty} dk_0 \frac{ie^{-ik_0 t} f(k_0)}{(k_0 - E_{\mathbf{k}} + i\epsilon)(k_0 + E_{\mathbf{k}} - i\epsilon)}. \quad (43)$$

Here  $t$  is the difference of the time variables introduced above between the two vertices connected by the line carrying momentum  $k_{\mu}$  in the same direction of  $t$ .  $f(k_0)$  is usually a polynomial of finite degree and analytic in the complex  $k_0$  plane. The above integral is evaluated using contours. For  $t > (<)0$ , one closes the contour in the lower (upper) half plane picking up the residue at  $k_0 = E_{\mathbf{k}} - i\epsilon$  ( $k_0 = -E_{\mathbf{k}} + i\epsilon$ ), with the result

$$\frac{\pi}{E_{\mathbf{k}}} \left[ \tau(t) e^{-i(E_{\mathbf{k}} - i\epsilon)t} f(+E_{\mathbf{k}}) + \tau(-t) e^{+i(E_{\mathbf{k}} - i\epsilon)t} f(-E_{\mathbf{k}}) \right]. \quad (44)$$

A crucial condition for the above manipulation is that  $f(k_0)$  must not blow up at the lower (upper) infinite semi-circle in the complex plane faster than  $e^{-ik_0 t}$  decays. Finally, one completes the time integrals sequentially and arrives at the result in the time-ordered perturbation theory. Now let us try to do the opposite in NC theory from

the time-ordered perturbation theory to the covariant one by turning around the above procedure. In this case the function  $f(\pm E_{\mathbf{k}})$  is an NC phase which is essentially an exponential (superposition) of the form  $\exp(\pm iE_{\mathbf{k}}\tilde{k}'^0)$ , where  $\tilde{k}'^0 = \theta_{0i}k'^i$  with  $\mathbf{k}'$  being a spatial momentum of some other internal or external line. If the above procedure were reversible, the corresponding function in the complex  $k_0$  plane would be something like  $\exp(ik_0\tilde{k}'^0)$  so that the naive result might have a chance to be recovered. But this is impossible because it is not guaranteed that it increases slower than  $e^{-ik_0t}$  decays. Actually whether it decays or blows up depends on the sign of  $\tilde{k}'^0$  which itself changes with  $\mathbf{k}'$ . This thus interferes with the above contour integration. One may argue that we may shift  $t$  to absorb  $\tilde{k}'^0$ . This is again not legitimate since  $t$  is a time difference and doing so simply spoils the time-ordering procedure which is a key bridge to relate the two formalisms. Furthermore, a connected diagram has certainly more than one line; the above shift of  $t$ 's, if it worked at all for one of them, would also interfere with each other making the trick totally useless. The above argument fails only if at tree level  $\tilde{k}'^0$  is an external momentum and happens to vanish. But this is a very special kinematic configuration if possible at all. We surely cannot rely on this in favour of the naive approach. From this analysis it is also clear that the main obstacle originates from the non-local exponential interaction that is intrinsic in NC field theory.

Finally we extend the above prescriptions to amputated and connected Green functions by adding the following rule concerning external lines for time-ordered diagrams. It should also be applied to each individual diagram in which the connection of external lines to vertices is fixed.

(6) Multiply by a factor of  $\eta_{\lambda_e}(p_e)$  for each external line with incoming momentum  $p_e$  and time direction parameter  $\lambda_e$  which is  $+$ ( $-$ ) if it connects to a(n) later (earlier) vertex. This same  $\lambda_e$  also appears in the preceding vertex factors (where it takes one of the signs for  $S$ -matrix). Sum over the set  $\{\lambda_e\}$ .

## 4 Discussion and conclusion

Based on the assumption that a time-ordered expansion of a formally unitary time evolution operator is a good starting point also on NC spacetime, we studied perturbative NC field theory which turns out to be the time-ordered perturbation theory adapted properly to the NC case. This was achieved by a detailed analysis of some exemplifying Green functions which we worked out with care, and then extended to the general case. We found no obstacles in implementing non-commutativity in perturbation theory whether time commutes with space or not. We provided prescriptions for computing  $S$ -matrix elements and amputated Green functions.

Although we only treated scalar fields in this paper, we expect no problems with spinor fields as is the case on ordinary spacetime. Since the spinor effect amounts to an additional numerator associated with a propagator, it

plays a similar role as a vertex, namely, the momentum contained therein will be on-shell with positive or negative energy. The situation is more complicated for gauge bosons. But again as in the usual theory there should be no problem at least in the 't Hooft–Feynman gauge. Our method also applies to any dimensions.

The NC perturbation theory thus obtained is already different at tree level from the naive approach followed in the literature. The interaction vertices involve only on-shell momenta of positive or negative energy of participating particles. The basic quantity connecting the vertices is not the causal Feynman propagator but the individual propagation functions of positive and negative frequency. These elements are naturally incorporated in the framework of the time-ordered perturbation theory. In contrast to the ordinary field theory, it seems impossible to recast the NC time-ordered theory into a covariant form as has been assumed in the naive approach. We attributed this difference to the highly non-local character of phase-type NC interactions which has a significant impact on the analyticity properties of Green functions in the complex energy plane.

Since the whole picture for perturbation theory has been changed, we expect some of the phenomena found previously in the naive approach will also be altered. Among them we would like to mention briefly the unitarity issue which we will detail soon in a separate paper. Since the NC phase now involves only on-shell momentum, it is independent of the zeroth component of a generally off-shell four-momentum. The analyticity properties of Green functions in the complex plane of the zeroth component will thus be very different from that in the naive approach. Furthermore, the right-hand side of the unitarity relation for Green functions is also modified due to the change of vertices, which does not seem to have been noticed thus far. Considering all of this it is quite reasonable to expect that unitarity will be practically preserved as it is formally built in the time-ordered perturbation theory. This may give some hints for the interplay of string and field theory.

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As we were preparing the manuscript a new preprint [14] appeared in which the ideas developed in [9] were further elaborated on by some examples.

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